Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 317 (2008) 190–198
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

A critical evaluation of closure methods via two simple dynamic systems

M. Grigoriu

Cornell University, Ithaca, NY 14853-3501, USA

Received 16 October 2007; received in revised form 18 February 2008; accepted 20 February 2008
Handling Editor: S. Bolton
Available online 18 April 2008

Abstract

Two simple dynamic systems with cubic nonlinearity and additive Gaussian white noise are used to assess the performance and the usefulness of closure methods in nonlinear random vibration. One of the systems has a single potential well while the other has two potential wells. It is shown that the performance of closure methods is determined by the structure of the moment equations rather than the way in which these equations are closed. For the system with one potential well, any closure method is satisfactory. For the system with two potential wells, closure methods can be inaccurate irrespective of the closure level. It is also shown that moment equations can be augmented with moment inequalities to solve approximately the infinite hierarchy of moment equations.

1. Introduction

Consider a dynamic system with state X driven by Gaussian white noise. The probability law of X is available analytically in few cases of limited practical interest. Monte Carlo estimates of the law of X are always possible but are likely to be impractical in realistic applications since the computation time required to generate samples of X can be excessive. Because of limitations of analytical and Monte Carlo solutions, perturbation, equivalent linearization, stochastic averaging, closure, and other approximate methods are commonly used in applications to calculate statistics of X [1, Section 7.3.1.5].

Closure methods involve two steps. First, equations are constructed for the moments of X. We note that moment equations exist if the nonlinear terms in the defining equation of X are polynomials of the system state. Second, a finite number $q \geqslant 1$ of equations, referred to as closure level, is selected from the infinite hierarchy of moment equations. Since the number of unknown moments of X in the selected moment equations exceeds q, it is not possible to find the moments of X exactly [2,3]. Closure methods postulate relationships between some of the moments of X that, together with the moment equations, provide the needed number of equalities to calculate the unknown moments of X in the moment equations up to a selected closure level [1, Section 7.3.1.5].

E-mail address: mdg12@cornell.edu

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2008.02.049

The state $X(t)$ of the dynamic systems considered in our discussion is the solution of the stochastic differential equation

$$\mathrm{d}X(t) = (\alpha X(t) + \beta X(t)^3)\,\mathrm{d}t + \sigma\,\mathrm{d}B(t), \quad t \geqslant 0, \tag{1}$$

where $\alpha \in \mathbb{R}$, $\beta < 0$, $\sigma \in \mathbb{R}$, and B denotes a standard Brownian motion process. We consider only the stationary solution of Eq. (1). Since $-X$ is a solution if X is a solution, the equality $X(t) \overset{\mathrm{d}}{=} -X(t)$ holds in distribution at any time t, so that the density of $X(t)$ is an even function. Accordingly, we need to calculate only the even order moments of $X(t)$ since the odd order moments of the system state are 0. We also note that the dynamic system in Eq. (1) has a single potential well for $\alpha < 0$ and two potential wells for $\alpha > 0$.

Our objective is to assess the performance and usefulness of closure methods in random vibration, and use Eq. (1) to achieve this objective. The paper extends results in Ref. [4] established for systems with a single potential well, that is $\alpha < 0$ in Eq. (1). We present an elementary proof of the main result in Ref. [4], which states that, if $\alpha < 0$, there exists a unique value of the stationary second moment $E[X(t)^2]$ of X such that all moments $E[X(t)^{2k}]$, $k = 2, 4, \ldots$, are positive, this unique value of $E[X(t)^2]$ is the exact value of the second moment of the system state, and any closure method provides satisfactory approximations for the moments of $X(t)$ if based on a sufficiently large closure level. This result is used to construct a sequence of intervals that decreases with the closure level and contains the exact value of $E[X(t)^2]$.

If $\alpha > 0$, that is, the system in Eq. (1) has two potential wells, the moment equations provide no constraint on the possible values of $E[X(t)^2]$ regardless of the closure level. Any $E[X(t)^2] > 0$ delivers positive moments $E[X(t)^{2k}]$, $k = 2, 4, \ldots$, so that it is an acceptable solution. Closure methods can be unsatisfactory in this case. It is also shown that moment inequalities can be used in conjunction with moment equations to construct a sequence of bounded intervals in $(0, \infty)$ that contain the exact value of the second moment of $X(t)$.

The results in the paper show that the performance of closure methods is determined by the structure of the moment equations rather than the way in which the infinite hierarchy of moment equations is closed, that is, the particular closure method used for solution. This finding is of great concern in applications since it is not possible to assess the performance of a particular closure method without extensive calculations, for example, large-scale Monte Carlo simulations, in which case closure methods may not be needed.

2. Properties of X

The Fokker–Planck equation,

$$\frac{\partial f(x,t)}{\partial t} = -\frac{\partial}{\partial x}((\alpha x + \beta x^3) f(x,t)) + \frac{\sigma^2}{2}\frac{\partial^2 f(x,t)}{\partial x^2}, \tag{2}$$

can be used to calculate the marginal density $f(x,t)$ of $X(t)$ in Eq. (1) at any time t [5]. Since the process $X(t)$ approaches stationarity as time increases indefinitely, its marginal density becomes time-invariant, that is, $\lim_{t\to\infty} f(x;t) = f(x)$, so that Eq. (2) degenerates into an ordinary differential equations with solution

$$f(x) = c\,\exp\left(\frac{1}{\sigma^2}\left(\alpha x^2 + \frac{1}{2}\beta x^4\right)\right), \quad x \in \mathbb{R}, \tag{3}$$

where $c > 0$ denotes a normalizing constant. Moments of any order p of the stationary solution $X(t)$ of Eq. (1) can be calculated from

$$\mu_p = \int_{\mathbb{R}} x^p f(x)\,\mathrm{d}x, \quad p = 1, 2, \ldots, \tag{4}$$

by numerical integration. Since $f(x) = f(-x)$, $x \in \mathbb{R}$, is an even function, the odd order moments of $X(t)$ vanish, so that we need to consider only even order moments.

If $\alpha, \beta < 0$, the stationary density $f(x)$ in Eq. (3) has a single mode centered at $x = 0$. If $\alpha > 0$ and $\beta < 0$, then $f(x)$ has two modes located at $x = \pm\sqrt{-\alpha/\beta}$ corresponding to the two wells of the potential of the dynamic system described by Eq. (1). Since the stationary density $f(x)$ exhibits a qualitative change at $\alpha = 0$, we say that it undergoes a transition [1, Section 9.4.3, 6, Section 6.3, 7]. The bifurcation point $\alpha = 0$ separates most likely value of X centered at $x = 0$ for $\alpha < 0$ from those centered at $x = \pm\sqrt{-\alpha/\beta}$ for $\alpha > 0$. We note that stationary moments of X cannot capture the transition of $f(x)$ at $\alpha = 0$ since they smear motion details associated with jumps between potential wells [6, Section 6.4, 7]. It is suggested that this limitation of the stationary moments of X causes failure of the closure method for $\alpha > 0$.

3. Moment equations

Let $\mu(p,t) = E[X(t)^p]$ denote the moment of order $p = 1,2,\ldots$ of $X(t)$ at an arbitrary time t. Itô's formula applied to the mapping $X(t) \mapsto X(t)^p$ gives

$$X(t)^p - X(0)^p = \int_0^t pX(t)^{p-1}\,\mathrm{d}X(t) + \frac{\sigma^2}{2}\int_0^t p(p-1)X(t)^{p-2}\,\mathrm{d}[B,B](t), \tag{5}$$

where $[B,B](t)$ denotes the quadratic covariation process of $B(t)$ [1, Section 7.3.1.1, 8, Sections 4.1 and 4.2]. The expectation of Eq. (5) followed by differentiation with respect to time gives

$$\dot{\mu}(t;p) = p\alpha\mu(t,p) + p\beta\mu(t,p+2) + \frac{p(p-1)\sigma^2}{2}\mu(t,p-2), \quad p = 1,2,\ldots, \tag{6}$$

with the convention $\mu(t,p) = 0$ for $p < 0$. Let $\mu_p = \lim_{t\to\infty}\mu(t,p)$ denote the stationary moment of order p of X. Since the moments of X are time-invariant during the stationary regime, Eq. (6) yields

$$p\alpha\mu_p + p\beta\mu_{p+2} + \frac{p(p-1)\sigma^2}{2}\mu_{p-2} = 0, \quad p = 1,2,\ldots, \tag{7}$$

in the limit as $t \to \infty$.

The odd order moments of X are 0, so that they satisfy Eq. (7) identically. The even order moments must satisfy the conditions

$$2k\alpha\mu_{2k} + 2k\beta\mu_{2(k+1)} + \frac{2k(2k-1)\sigma^2}{2}\mu_{2(k-1)} = 0, \quad k = 1,2,\ldots, \tag{8}$$

giving the recurrence formulas

$$\mu_{2(k+1)} = a\mu_{2k} + (2k-1)b\mu_{2(k-1)}, \quad k = 1,2,\ldots, \tag{9}$$

where $a = -\alpha/\beta$ and $b = -\sigma^2/(2\beta)$. These equations are

$$\begin{aligned}\mu_4 &= a\mu_2 + b,\\ \mu_6 &= a\mu_4 + 3b\mu_2,\\ \mu_8 &= a\mu_6 + 5b\mu_4,\end{aligned} \tag{10}$$

for $k = 1,2,3$ and $\mu_2 > 0$, so that

$$\begin{aligned}\mu_4 &= a\mu_2 + b,\\ \mu_6 &= (a^2+3b)\mu_2 + ab,\\ \mu_8 &= (a^3+8ab)\mu_2 + a^2b + 5b^2,\end{aligned} \tag{11}$$

showing that all even order moments of X depend linearly on μ_2.

3.1. Closure methods

If the system in Eq. (1) is linear, that is, $\beta = 0$, then Eq. (7) becomes

$$p\alpha\mu_p + \frac{p(p-1)\sigma^2}{2}\mu_{p-2} = 0, \quad p = 1,2,\ldots, \tag{12}$$

so that the moments of $X(t)$ can be calculated recursively since $\mu_0 = 1$. If $\beta < 0$, the moments of $X(t)$ satisfy Eq. (9), and its solution requires knowledge of both μ_0 and μ_2. Since μ_2 is not known, it is not possible to calculate the moments of $X(t)$ exactly.

Closure methods extract a finite set of equalities from Eq. (9) and provide an additional relationship between some of the moments of $X(t)$. Consider the first q equalities in Eq. (9), that is, the equalities in this equation for $k = 1, \ldots, q$, where $q \geqslant 1$ is an integer referred to as closure level. The resulting system of q equations has $q+1$ unknowns, the moments $\mu_2, \ldots, \mu_{2(q+1)}$, so that it cannot be solved. For example, at closure level $q = 1$, the system of moment equations is $\mu_4 = a\mu_2 + b$, a single equation with two unknowns, the moments μ_2 and μ_4. An additional relationship between μ_2 and μ_4 is needed for solution, and this relationship is provided by closure methods.

For example, according to the Gaussian closure method applied at a closure level q, the moments μ_{2q} and $\mu_{2(q+1)}$ relate in the same way as the corresponding moments of Gaussian random variables [2]. Let $G \sim N(0, \gamma)$ be a Gaussian variable with mean 0 and variance γ. Since the moments of G of order $2r$, $r = 1, 2, \ldots$, are

$$E[X^{2r}] = \frac{(2r)!\gamma^r}{2^r r!}, \tag{13}$$

we have

$$E[X^{2(r+1)}] = (2r+1)\gamma E[X^{2r}]. \tag{14}$$

Under the assumption that the relationship between μ_{2q} and $\mu_{2(q+1)}$ matches that in Eq. (14), that is, $\mu_{2(q+1)} = (2q+1)\gamma\mu_{2q} = (2q+1)\mu_2\mu_{2q}$, it is possible to solve the system of q moment equations and find approximate values for the moments $\mu_2, \ldots, \mu_{2(q+1)}$ of $X(t)$. If $q = 1$, we have $\mu_4 = 3\gamma\mu_2 = 3\mu_2^2$, so that, at this closure level, μ_2 is the solution of

$$3\mu_2^2 = a\mu_2 + b. \tag{15}$$

The positive roots of Eq. (15) are $\mu_2 = 0.2743$ for $(\alpha = -1, \beta = -1, \sigma = 1)$ and $\mu_2 = 0.6076$ for $(\alpha = 1, \beta = -1, \sigma = 1)$. The errors of the approximate second-order moments of $X(t)$ are -5 and -32 percent, respectively. As it will be shown in the following sections, moment closure solutions improve with q for the system with a single potential well but may not improve or yield negative even order moments for the systems with two potential wells.

3.2. Range of μ_2

Let

$$I_k = \{\mu_2 > 0 : \mu_{2(k+1)} > 0\}, \quad k = 1, 2, \ldots \tag{16}$$

denote the range of μ_2 in $(0, \infty)$ such that the moment $\mu_{2(k+1)}$ is positive. Then the interval $I^{(q)} = \bigcap_{k=1}^{q} I_k$ contains the values of μ_2 such that the moments $\mu_{2(k+1)}$, $k = 1, \ldots, q$, entering the moment equations up to a closure level q are positive. It is shown that the intervals $I^{(q)}$ decrease with q and are q-invariant for $\alpha < 0$ and $\alpha > 0$, respectively. In the latter case, the moment equations only tell us that μ_2 must be positive, so that they provide no information on the range of values of μ_2.

3.2.1. Case 1: $\alpha < 0$

Let $\alpha = -1$, $\beta = -1$, and $\sigma = 1$. Fig. 1 shows the dependence of moments μ_{2k}, $k = 2, \ldots, 6$, on μ_2 in the range $(0.1, 0.5)$. The horizontal heavy segment in the figure is the interval $I^{(6)} = (0.257, 0.3)$. As expected, the exact solution $\mu_2 = 0.2896$ is included in $I^{(6)}$.

The intervals I_k are $I_1 = (0.0, 0.5)$, $I_2 = (0.2, \infty)$, $I_3 = (0.0, 0.35)$, $I_4 = (0.25, \infty)$, $I_5 = (0.0, 0.3)$, and $I_6 = (0.257, \infty)$, for $k = 1$, 2, 3, 4, 5, and 6, respectively. The corresponding intervals $I^{(q)}$ are $I^{(1)} = (0.0, 0.5)$, $I^{(2)} = (0.2, 0.5)$, $I^{(3)} = (0.2, 0.35)$, $I^{(4)} = (0.2, 0.35)$, $I^{(5)} = (0.2, 0.3)$, and $I^{(6)} = (0.257, 0.3)$, and form a decreasing sequence. According to results in Ref. [4], the decreasing sequence of intervals $\{I^{(q)}\}$ has a unique limit, and this limit is the exact value of μ_2. Moreover, the sequence $\{\mu_2(q)\}$, $q = 1, 2, \ldots$, of approximate

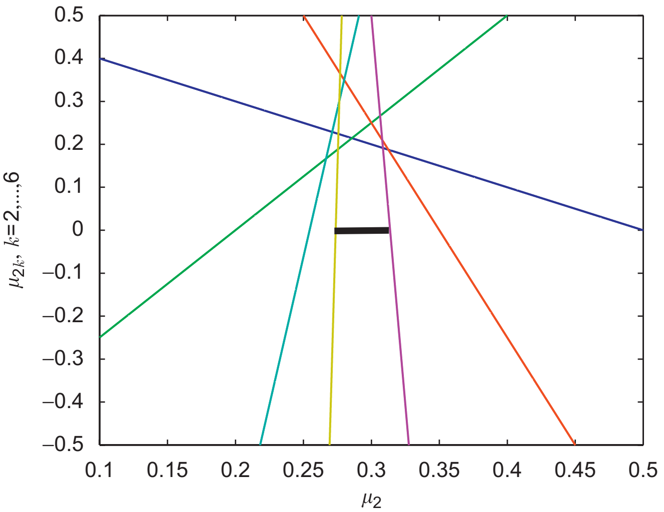

Fig. 1. Relationship between μ_{2k}, $k = 2, \ldots, 7$ and μ_2 for $\alpha = -1$, $\beta = -1$, and $\sigma = 1$. The horizontal heavy segment marks the interval $I^{(6)}$.

values of μ_2 corresponding to increasing closure levels also converge to the exact value of μ_2, that is, $\lim_{q\to\infty}\mu_2(q) = \mu_2$, irrespective of the particular closure method used for calculations.

That the last statement holds results from Fig. 1. Consider the system of moment equations at a closure level q and an arbitrary closure condition $\mu_{2(q+1)} = \zeta\mu_{2q}$, $\zeta > 0$, relating the two highest-order moments. We now have $q+1$ unknowns and as many equations so that the moments $\mu_2, \ldots, \mu_{2(q+1)}$ of $X(t)$ can be calculated. For $q = 1$, we have $\mu_4 = -\mu_2 + \frac{1}{2}$ and $\mu_4 = \zeta\mu_2$, so that $\mu_2 = (\frac{1}{2})/(\zeta+1) \in I^{(1)}$. Similarly, at closure level $q = 2$, we have $\mu_6 = (\frac{5}{2})\mu_2 - \frac{1}{2}$ and $\mu_6 = \zeta\mu_4 = -\zeta\mu_2 + \zeta/2$, so that $\mu_2 = (\zeta+1)/(2\zeta+5) \in I^{(2)}$, and so on. This shows that, for any $\zeta > 0$, that is, any closure method, the approximate value of μ_2 belongs to $I^{(q)}$ at each closure level $q = 1, 2, \ldots$, and since $\lim_{q\to\infty} I^{(q)} = \{\mu_2\}$, any closure methods is asymptotically correct as $q \to \infty$; it does not matter how the moment equations are closed. If the closure level is sufficiently high, any closure methods provides satisfactory approximations for the moments of $X(t)$. Claims that a closure method is superior to another are unfounded. The success of closure methods relates solely to the structure of the moment equations.

3.2.2. Case 2: $\alpha > 0$

Let $\alpha = 1$, $\beta = -1$, and $\sigma = 1$. Fig. 2 shows the variation of the moments μ_{2k}, $k = 2, \ldots, 6$, with μ_2 in the range $(0.0, 0.5)$. The moments have been calculated from Eq. (9) sequentially. The plots show that any $\mu_2 > 0$ delivers positive moments μ_{2k}, $k = 2, \ldots, 6$, so that it is an acceptable solution at closure level $q = 6$.

It was shown that, if $\alpha < 0$, the sequence of intervals $\{I^{(q)}\}$, $q = 1, 2, \ldots$, decreases with q and approaches the exact values of the second moment of $X(t)$, that is, $\lim_{q\to\infty} I^{(q)} = \{\mu_2\}$. In the case $\alpha > 0$ considered here all intervals I_k are equal to $(0, \infty)$ since the coefficients $a = -\alpha/\beta = 1$ and $b = -\sigma^2/(2\beta) = \frac{1}{2}$ are positive, so that the moments of order 4 and higher of $X(t)$ are positive for $\mu_2 > 0$. Accordingly, we have $I^{(q)} = (0, \infty)$ for all closure levels q. The moment equations provide no useful information; they only tell us that any $\mu_2 > 0$ is a feasible solution.

The performance of closure methods applied to the dynamic systems with $\alpha < 0$ and $\alpha > 0$ differ significantly. We have seen that for $\alpha < 0$ any closure method is asymptotically correct as $q \to \infty$. In the case $\alpha > 0$ the situation is quite different. Consider as previously an arbitrary closure method postulating the relationship $\mu_{2(q+1)} = \zeta\mu_{2q}$ for some $\zeta > 0$. At closure level $q = 1$, we have $\mu_4 = \mu_2 + \frac{1}{2}$ and $\mu_4 = \zeta\mu_2$, so that $\mu_2 = (\frac{1}{2})/(\zeta - 1) \in I^{(1)} = (0, \infty)$ for $\zeta \in (0, 1)$. There is no solution for $\zeta > 1$, in the sense that the resulting value of μ_2 is negative. At closure level $q = 2$, we have $\mu_6 = (\frac{5}{2})\mu_2 + \frac{1}{2}$ and $\mu_6 = \zeta\mu_4 = \zeta\mu_2 + \zeta/2$ so that $\mu_2 = (1-\zeta)/(2\zeta - 5) \in I^{(2)} = (0, \infty)$ for $\zeta \in (1, \frac{5}{2})$. There is no solution for$\zeta \in (1, \frac{5}{2})^c$, in the sense that the resulting value of μ_2 is negative.

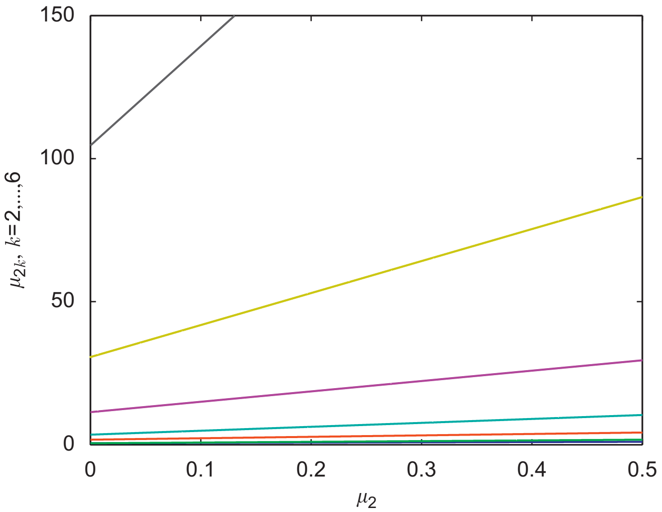

Fig. 2. Relationship between μ_{2k}, $k = 2, \ldots, 7$ and μ_2 for $\alpha = 1$, $\beta = -1$, and $\sigma = 1$.

Since (1) the approximate value of μ_2 is sensitive to the value of ζ, that is, the particular closure method considered for solution, (2) the selection of ζ is based on heuristic arguments, and it is not possible to identify an optimal value for this parameter, (3) moments of order 4 and higher of $X(t)$ depend strongly on the value of μ_2 (Fig. 2), and (4) resulting approximations of μ_2 are negative for some values of ζ, that is, for some closure methods, we conclude that, if $\alpha > 0$, closure methods are unsatisfactory. The performance of closure methods in this case is markedly different from that for $\alpha < 0$, and shows that the performance of closure methods is determined by the structure of the moment equations rather than the closure technique. This is a serious limitation since it is not possible to assess the accuracy of a particular closure method for a particular dynamic system without, for example, extensive Monte Carlo simulations, in which case the use of closure methods will not be justified.

4. Moment inequalities

Let Y and Z be two real-valued random variables with finite variance. According to the Cauchy–Schwarz inequality, we have $|E[YZ]| \leqslant (E[Y^2]E[Z^2])^{1/2}$. This inequality applied to the even powers of the stationary solution $X(t)$ of Eq. (1) gives

$$E[X(t)^{2r}] = E[X(t)^{2r-p}X(t)^p] \leqslant (E[X(t)^{2(2r-p)}]E[X(t)^{2p}])^{1/2} \tag{17}$$

or

$$\mu_{2r} \leqslant (\mu_{2(2r-p)}\mu_{2p})^{1/2}. \tag{18}$$

The inequalities in Eq. (18) provide distinct conditions only for $p = 0, 1, \ldots, k-1$ because of their symmetry. These conditions are

$$\begin{aligned}
r = 1: &\quad \mu_2 \leqslant \mu_4^{1/2};\\
r = 2: &\quad \mu_4 \leqslant \mu_8^{1/2},\ \ \mu_4 \leqslant (\mu_6\mu_2)^{1/2};\\
r = 3: &\quad \mu_6 \leqslant \mu_{12}^{1/2},\ \ \mu_6 \leqslant (\mu_{10}\mu_2)^{1/2},\ \ \mu_6 \leqslant \mu_8\mu_4)^{1/2};\\
r = 4: &\quad \mu_8 \leqslant \mu_{16}^{1/2},\ \ \mu_8 \leqslant (\mu_{14}\mu_2)^{1/2},\ \ \mu_8 \leqslant (\mu_{12}\mu_4)^{1/2},\ \ \mu_8 \leqslant (\mu_{10}\mu_6)^{1/2};
\end{aligned} \tag{19}$$

for $r = 1, \ldots, 4$.

4.1. Range of μ_2

The inequalities in Eq. (18) provide additional constraints on the moments of $X(t)$ that, together with the moment equations, can be used to construct tighter bounds on the possible values of μ_2. Let

$$J_k = \{\mu_2 > 0 : \mu_{2(k+1)} > 0 \text{ and inequalities in Eq. (9)}\}, \quad k = 1, 2, \ldots \tag{20}$$

denote the range of μ_2 in $(0, \infty)$ such that $\mu_{2(k+1)}$ is positive and satisfies the appropriate conditions in Eq. (19). For example, the inequality constraint for J_1 is $\mu_2 \leqslant \mu_4^{1/2}$. Since J_k includes all constraints in the definition of I_k, we have $J_k \subset I_k$. Let $J^{(q)} = \bigcap_{k=1}^{q} J_k$ be the interval containing $\mu_2 > 0$ with the property that the higher-order moments of $X(t)$ in the equations up to closure level q are positive and satisfy inequalities of the type in Eq. (19). Since $J_k \subset I_k$, we have $J^{(q)} \subset I^{(q)}$.

The construction of the intervals J_k and $J^{(q)}$ involves two steps. First, Eq. (9) is used to express the moments of $X(t)$ of order 4 and higher in the moment equations up to closure level q in terms of μ_2. Second, the resulting moments are introduced in Eq. (18) to obtain additional constraints on the possible values of μ_2.

4.1.1. Case 1: $\alpha < 0$

We have seen that the moment equations deliver intervals $I^{(q)}$, $q = 1, 2, \ldots$, containing values of μ_2 for which the moments of order $4, \ldots, 2(q+1)$ of $X(t)$ are positive. The intervals $\{I^{(q)}\}$ constitute a decreasing sequence converging to the unique solution μ_2 of the moment equations. It was also found that $I^{(6)} = (0.257, 0.3)$ for $\alpha = -1$, $\beta = -1$, and $\sigma = 1$, that is, if $\mu_2 \in I^{(6)}$, the moments of $X(t)$ up to order 14 calculated from Eq. (10) are positive.

The intervals J_k obtained from the moment equations and the first, second, third, and fourth inequalities in Eq. (19) are $(0.0, 0.3660)$, $(0.2745, 0.3450)$, $(0.2615, 0.2945)$, and $(0.2875, 0.3005)$, respectively, so that

$$J^{(1)} = (0.0, 0.3660),$$

$$J^{(2)} = (0.2745, 0.3450),$$

$$J^{(3)} = (0.2745, 0.2945), \quad \text{and}$$

$$J^{(4)} = (0.2875, 0.2945).$$

Hence, at closure level $q = 4$, the moment equations and the moment inequalities are satisfied if μ_2 belongs $J^{(4)} = (0.2875, 0.2945)$. The range $J^{(4)}$ of possible values of μ_2 based on both moment equations and inequalities is tighter than the interval $I^{(6)} = (0.257, 0.3)$ based on a higher closure level but only moment equations, that is, $J^{(4)} \subset I^{(6)}$. Since $I^{(q)}$ is a decreasing sequence of intervals, $J^{(4)} \subset I^{(4)}$ also holds.

We note that $J^{(4)} = (0.2875, 0.2945)$ is a tight interval that includes the exact value $\mu_2 = 0.2896$ of the second moment of $X(t)$. Any value in $J^{(4)} = (0.2875, 0.2945)$ is a satisfactory approximation of μ_2 that yields accurate values for the moments of order 4 and higher of $X(t)$.

4.1.2. Case 2: $\alpha > 0$

It was shown that the intervals $I^{(q)}$, $q = 1, 2, \ldots$, containing the exact value of μ_2 are equal to $(0, \infty)$ for all closure levels q. Hence, the moment equations alone provide no information on the exact value of μ_2 irrespective of the closure level. However, moment equations augmented with moment inequalities can be used to construct meaningful bounds on the exact value of μ_2. For example, values of μ_2 satisfying the first, second, third, and fourth inequalities in Eq. (19) are contained in the intervals $(0.0, 1.3660)$, $(0.6077, 4.3452)$, $(0.1277, 1.6950)$, and $(0.4875, \infty)$, respectively, so that

$$J^{(1)} = (0.0, 1.3660),$$

$$J^{(2)} = (0.6077, 1.3660),$$

$$J^{(3)} = (0.6077, 1.3660), \quad \text{and}$$

$$J^{(4)} = (0.6077, 1.3660).$$

We note that (1) the exact value $\mu_2 = 0.8935$ of the second moment of $X(t)$ is included in $J^{(4)}$, (2) increasing the closure level to $q = 8$ does not produce intervals tighter than $J^{(4)}$, so that the bounds on $\mu_2 = 0.8935$ remains relatively wide, and (3) $J^{(4)}$ for $\alpha = 1$ is much wider than $J^{(4)}$ for $\alpha = -1$.

5. Moment distributions

We have found from moment equations and moment inequalities that the feasible values of μ_2 belong to intervals $J^{(q)}$ depending on closure level q. The recurrence formula in Eq. (9) and moment inequalities in Eq. (18) can be used to construct the intervals $J^{(q)}$. The range of values of the moments of $X(t)$ of order 4 and higher corresponding to $J^{(q)}$ can be obtained from the recurrence formulas in Eqs. (9)–(11). For example, if $\alpha = -1$, $\beta = -1$, and $\sigma = 1$, the range $J^{(q)} = (0.2875, 0.2945)$ of μ_2 at $q = 4$ is mapped into the intervals $(0.2055, 0.2125)$, $(0.2188, 0.2388)$, and $(0.2775, 0.3125)$ for μ_4, μ_6, and μ_8, respectively. These intervals include the exact values $(0.2104, 0.2240, 0.3020)$ of (μ_4, μ_6, μ_8). If $\alpha = 1$, $\beta = -1$, and $\sigma = 1$, $J^{(q)} = (0.6077, 1.3660)$ at $q = 4$ is mapped into the intervals $(1.1077, 1.8660)$, $(2.0193, 3.9150)$, and $(4.7885, 8.5800)$ for μ_4, μ_6, and μ_8, and these intervals include the exact values $(1.3935, 2.7377, 6.2173)$ of (μ_4, μ_6, μ_8).

If the uncertain value of μ_2 is assumed to be a random variable with distribution F_2 of support $J^{(q)}$, a characterization of uncertain parameters used by Bayesian statisticians, the higher-order moments of $X(t)$ are also random variables with distributions $F_{2k}(\xi) = F_2((\xi - u)/v)$, where u and v denote translation and scale parameters. For example, the distributions of μ_4, μ_6, and μ_8 are, respectively,

$$F_4(\xi) = F_2\left(\frac{\xi - b}{a}\right),$$

$$F_6(\xi) = F_2\left(\frac{\xi - ab}{a^2 + 3b}\right),$$

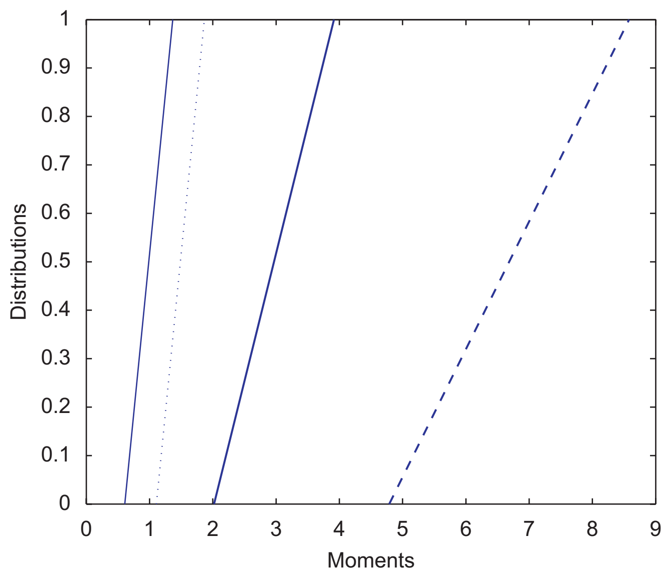

Fig. 3. Distributions of μ_2, μ_4, μ_6, and μ_8 in thin solid, thin dashed, heavy solid, and heavy dashed lines, respectively. Results are for $\alpha = 1$, $\beta = -1$, and $\sigma = 1$.

$$F_8(\xi) = F_2\left(\frac{\xi - a^2b - 5b^2b}{a^3 + 8ab}\right). \tag{21}$$

The supports of these distributions coincide with the moment ranges established in the previous paragraph. Fig. 3 shows the distributions of μ_2, μ_4, μ_6, and μ_8 with thin solid, thin dashed, heavy solid, and heavy dashed lines for $\alpha = 1$, $\beta = -1$, and $\sigma = 1$ under the assumption that μ_2 is uniformly distributed in $J^{(4)}$. The distributions of μ_4, μ_6, and μ_8 are translated and scaled versions of the distribution of μ_2. The random variables μ_{2k}, $k = 2, 3, \ldots$, have the representations $\mu_{2k} = a_{2k}\mu_2 + b_{2k}$ depending on some coefficients (a_{2k}, b_{2k}), so that the correlation coefficients $\rho_{k,l} = a_{2k}a_{2l}/(a_{2k}^2 a_{2l}^2)^{1/2}$ between μ_{2k} and μ_{2l}, $k \neq l$, are $+1$ or -1, that is, these random variables are positively or negatively perfectly correlated.

We also note that, if information beyond $\mu_2 \in J^{(q)}$ becomes available, the (noninformative) uniform density with support $J^{(q)}$ used for μ_2 can be updated within a Bayesian framework [9]. The additional information may consist of a relatively short sample of $X(t)$, that is sufficient to estimate μ_2 but not the higher-order moments of $X(t)$. Considerations of this type are beyond the objective of this paper.

6. Conclusions

Analytical expressions for the moments and the distributions of the state X of nonlinear dynamic systems driven by Gaussian white noise are only available in simple cases of limited practical interest. Monte Carlo simulation can be used to estimate properties of X provided that the required computation time is not excessive. Properties of X can also be obtained by approximate methods, for example, equivalent linearization, perturbation, stochastic averaging, closure methods. Our objective is to assess the performance and the usefulness of closure methods since these methods have been and are applied extensively in nonlinear random vibration.

Two simple dynamic systems with cubic nonlinearity and additive Gaussian white noise have been used to evaluate closure methods. One of the systems has a single potential well while the other has two potential wells. It was shown that the performance of closure methods is system dependent, in the sense that it is determined by the structure of the moment equations rather than the particular closure technique used to close these equations. This is a highly undesirable feature in applications. For the system with a single potential well, any closure method provides accurate results if based on a sufficiently large closure level. For the system with two potential wells, even order moments delivered by closure methods can be inaccurate or even negative. It was also shown that bounds can be established on the moments of X by using both moment equations and moment inequalities. The bounds are tight for the system with a single well and relatively wide for the system with two potential wells.

References

[1] M. Grigoriu, *Stochastic Calculus, Applications in Science and Engineering*, Birkhäuser, Boston, 2002.
[2] V.V. Bolotin, *Random Vibration of Elastic Systems*, Martinus Nijhoff Publishers, Boston, 1984.
[3] R.A. Ibrahim, *Parametric Random Vibration*, Research Studies Press, New York, 1984.
[4] A.M. Hasofer, M. Grigoriu, A new perspective on the moment closure method, *Journal of Applied Mechanics* 62 (2) (1995) 527–532.
[5] Y.K. Lin, *Probabilistic Theory of Structural Dynamics*, Robert E. Krieger Publishing Company, Huntington, New York, 1976.
[6] W. Horsthemke, R. Lefever, *Noise-Induced Transitions*, Springer, New York, 1984.
[7] R.A. Ibrahim, Excitation-induced stability and phase transitions: a review, *Journal of Vibration and Control* 12 (10) (2006) 1093–1170.
[8] B. Øksendal, *Stochastic Differential Equations. An Introduction with Applications*, Springer, New York, 1998.
[9] A. Zellner, *An Introduction to Bayesian Inference in Econometrics*, Wiley, New York, 1971.